

TOWARD CHARACTERIZING BOOLEAN TRANSFORMATIONS

BY

ALAN E. GELFAND

TECHNICAL REPORT NO. 21

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DEPARTMENT OF STATISTICS

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ABSTRACT

Binary switching nets have been presented as useful models for a variety of complex phenomena. Determinate binary functions describe the response of an element in such a net to its inputs. Such functions are called Boolean transformations. We study three structural properties of these transformations--forcibility, internal homogeneity and threshold. We have argued in two previous reports that these properties significantly describe the behavior of these transformations and of the switching net model itself.

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Alan E. Gelfand

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1. INTRODUCTION

Examining the behavior of binary switching net models has been offered as a useful tool for enhancing our understanding of the behavior of a variety of complex phenomena. Two such contexts are the development of a biological analogy by Kaufman (1970) applicable to genetic control systems and of an organizational analogy by Walker and Gelfand (1977) and Gelfand and Walker (1977) useful for formulating managerial strategy in complex organizations.

The objects of interest are networks of a finite number N of logic elements. Each element has k binary inputs and computes a specific, determinate binary function of those inputs at net time t . Each input is connected to some element's output, according to the particular net's fixed, unchanging structure, which is typically unknown. At net time $t+1$ the former output values become the input values on the appropriate elements, and the process repeats under these possibly new conditions. If by the state of the net at time t

we understand the ordered set of N element outputs at time t , the action of the net is to move in a determinate fashion from one net state to another as time proceeds. There are clearly 2^N distinct net states. Therefore from some arbitrary initial net state the net eventually must encounter a state it had shown before. Doing so, it must thereafter repeat the intermediate sequence of states. Such a sequence of states is called a cycle. If the number of distinct net states in a cycle is called the cycle length, it is apparent that cycle lengths may range from one to 2^N .

The determinate binary function on k inputs we refer to as a Boolean transformation or mapping. If all elements in a net are governed by the same transformation we refer to the net as homogeneous, otherwise heterogeneous. For convenience a mapping is usually presented in a table in lexicographic order as illustrated in Figure 1.

Thus a mapping on k inputs requires specification of its value for each of its 2^k possible input vectors and moreover there are a total of 2^{2^k} possible mappings. It may be convenient at times to denote a mapping m as a function of its inputs, i.e., $m(x_1, x_2, \dots, x_k)$.

As an ultimate goal we would like to be able to completely characterize any mapping in terms of a set of basic structural properties. Toward this objective we examine three significant structural properties,

x_3	x_2	x_1	m
1	1	1	e_1
0	1	1	e_2
1	0	1	e_3
0	0	1	e_4
1	1	0	e_5
0	1	0	e_6
1	0	0	e_7
0	0	0	e_8

Figure 1: General Representation of a Transformation on Three Inputs

forcibility, internal homogeneity and extended thresholds and their interrelationships. Forcibility was developed by Kaufman in a series of articles beginning in 1968. Internal homogeneity, a rather "natural" property is considered in some detail by Walker and Ashby (1966). Thresholds are discussed by Rosen (1958) and extended by the author in the present article. In Sections 2 and 3 we treat these properties.

It is hoped that these three properties in conjunction with other symmetries will provide a set of structural constraints which enable a unique representation of any mapping. Although, as we shall see, the given three properties do not uniquely determine a mapping; they do determine an equivalence class of

mappings across which behavior will be very similar. In many modeling instances this degree of explanation may suffice.

The general idea of attempting to factor a mapping by structural properties is also of interest. Such factorizations are discussed by Rosen (1958) and Babcock (1976). In Section 4 we correct and extend ideas considered in Babcock's thesis by achieving a unique factorization of a mapping through its forcing inputs.

2. DEFINITIONS

Let us formally define the structural features we will be discussing. A mapping is forcible on a given input when a given state of the input "forces" the output of the mapping to a single value regardless of the values of the other inputs. This given state is called the forcing state. If an input is forcing on both states then the mapping is either constant (trivial) or has half "1"'s and half "0"'s. In the former case all inputs are forcing on both states while in the latter case the mapping must be forcing only on the one input. Since forcibility with only one input is trivial we restrict attention to the case where the number of inputs $k \geq 2$. The forced value of an element is that value to which it is forcible.

If an element is forcible on more than one input line, its forced value is identical for all the inputs

on which it is forcible. This is apparent since forcibility on a particular input implies that the mapping assumes a forced value on at least 2^{k-1} of the 2^k input vectors. It is easy to verify that of the 16 mappings on two input coordinates ten are forcible on both coordinates (including the two trivial constant mappings), four are forcible on one and two on neither.

Internal homogeneity, denoted henceforth by I is the larger of the number of entries of "0" and of "1" in the table of values of a mapping, i.e., $I = \max(\#0's, \#1's)$ and hence $2^{k-1} \leq I \leq 2^k$. We denote by $N_k(i)$ the number of mappings with $I=i$.

We wish finally to define the notion of a mapping on k inputs which has threshold ℓ ($1 \leq \ell \leq k$). As we shall see, there is some confusion in formalizing such a definition. Customarily a system comprised of k switches is interpreted as having threshold ℓ if the system is "on" whenever at least ℓ switches are "on". In terms of our transformations we may say that a mapping on k inputs has threshold ℓ if whenever ℓ or more inputs take on a specified value the mapping takes on a specified value. The specified input value may be "0" or "1" and may be coupled with a mapping value of "0" or "1". Presumably in this definition ℓ is the minimum number of inputs for which the statement is true since if the statement holds at ℓ it would obviously

hold at $\ell+1, \ell+2, \dots, k$. The problems in formalizing such a definition may now be revealed. One difficulty centers around whether the definition is an if and only if statement. Can the system ever be "on" if fewer than ℓ inputs are "on"? If the answer is no we shall refer to ℓ as an absolute threshold but if the answer is yes we shall refer to ℓ only as a threshold. Every mapping must have a threshold (at the largest it would be k) but only a subset of mappings have an absolute threshold. A second difficulty arises because our more general conception of a threshold allows either "off" or "on" inputs to turn an element again either "off" or "on". How do we then assign a threshold value, ℓ to a mapping? We need to consider a threshold for the number of "off" or "0" inputs which we denote by $\ell_0(m)$ and similarly a threshold for the number of "on" or "1" inputs which we denote by $\ell_1(m)$. We then define $\ell(m) = \min(\ell_0(m), \ell_1(m))$. In light of our extended definition the minimum of these two numbers is clearly the more significant value.

For a fixed number of inputs, k , specifying the internal homogeneity, I , the threshold ℓ and the number of forcing inputs, j , defines equivalence classes within the set of 2^{2^k} possible mappings.

Figure 2 below shows that these three properties do not uniquely determine a mapping. With $k=3$ both mappings m_1 and m_2 have $I=7$, $\ell=2$ and $j=3$.

x_3	x_2	x_1	m_1	m_2
1	1	1	1	1
0	1	1	1	1
1	0	1	1	1
0	0	1	1	1
1	1	0	1	1
0	1	0	1	0
1	0	0	0	1
0	0	0	1	1

Figure 2: Two Members of the Equivalence Class
 $I=7$, $\ell=2$ and $j=3$.

3. THEORETICAL RESULTS

We will now state a few theoretical results which for a fixed number of inputs, k , detail the incidence of these properties, individually and jointly, amongst all mappings on k inputs. Proofs of these results are given in Walker and Gelfand (1977) or Gelfand and Walker (1977).

Recalling our notation of the previous section we have

Theorem 1: $N_k(i) = 2 \binom{2^k}{i}$ for $2^{k-1} < i \leq 2^k$ and if

$I = 2^{k-1}$, $N_k(2^{k-1}) = \binom{2^k}{2^{k-1}}$. Note: Henceforth, the

symbol $\binom{n}{r}$, n and r integers, denotes the number of combinations of n distinct objects taken k at a time,

$$\text{i.e., } \binom{n}{r} = \frac{n!}{r!(n-r)!}.$$

If $\Gamma(k, i, j)$ is defined to be the number of mappings on k inputs with internal homogeneity i , which are forcing on exactly j of the inputs we can show

Theorem 2: If $2^{k-1} < i < 2^k$ and $j > 0$

$$\Gamma(k, i, j) = \binom{k}{j} 2^{j+1} \gamma(k, i, j) \text{ where}$$

$$\gamma(k, i, j) = \begin{cases} \frac{2^{k-j}}{2^{k-1}} & \text{if } i < 2^k - 2^{k-j} + 2^{k-j-1} \\ \frac{\Gamma(k-j, i - 2^k + 2^{k-j}, 0)}{2} & \text{if } i > 2^k - 2^{k-j} + 2^{k-j-1} \\ \frac{2^{k-j}}{2^{k-j-1}} - 2(k-j) & \text{if } i = 2^k - 2^{k-j} + 2^{k-j-1} \end{cases}$$

and for the two extreme cases

Theorem 2':

$$(i) \quad \Gamma(k, 2^k, j) = 2, \quad j=k$$

$$= 0, \quad j \neq k$$

$$(ii) \quad \Gamma(k, 2^{k-1}, j) = 2k, \quad j=1$$

$$= \binom{2^k}{2^{k-1}} - 2k, \quad j=0$$

$$= 0, \text{ otherwise.}$$

From $\Gamma(k,i,j)$ we can calculate the number of mappings on k inputs with internal homogeneity i , which are forcing on at least (at most, etc.,...) j inputs by simple summation. Moreover we can calculate the number of mappings on k inputs forcing on exactly j inputs, $\Gamma(k,j)$ again by summation, i.e.,

$$\Gamma(k,j) = \sum_{i=2^{k-1}}^{2^k} \Gamma(k,i,j).$$

If we seek the density of any of these collections of mappings we standardize by either $N_k(i)$ or 2^{2^k} depending on the base of reference.

With regard to thresholds, let $\beta(k,\ell)$ be the number of mappings on k inputs with threshold ℓ (including those with absolute threshold ℓ). In computing $\beta(k,\ell)$ it will be convenient if we first calculate $\alpha(k,j,j')$ which is the number of mappings on k inputs with $\ell_0=j$ and $\ell_1=j'$. We have the following result.

Theorem 3:

$$(i) \alpha(k,j,j') = 2 \quad \text{if } j+j' = k+1$$

$$(ii) \alpha(k,j,j') = 2^{\binom{k}{j-1}+2} - 6 \quad \text{if } j+j' = k+2$$

$$(iii) \alpha(k,j,j') = 4(2^{\binom{k}{k-j+1}-1})(2^{\binom{k}{j-1}-1})2^{\sum_{i=k-j'+2}^{j-2} \binom{k}{i}} \\ \text{if } j+j' \geq k+3$$

from which we obtain $\beta(k, \ell)$ via

Theorem 4:

$$\beta(k, \ell) = 2 \sum_{\ell_1 = \max(\ell, k-\ell+1)}^k \alpha(k, \ell, \ell_1).$$

We next calculate $\tau(k, i, \ell)$, the number of mappings on k inputs having internal homogeneity i and threshold ℓ . It will be convenient as in Theorem 1 to calculate $\tau(k, i, \ell_0, \ell_1)$ first. Also for convenience let

$$c = \max\left(\sum_{j=\ell_0}^k \binom{k}{j}, \sum_{j=\ell_1}^k \binom{k}{j}\right), \quad d = \min\left(\sum_{j=\ell_0}^k \binom{k}{j}, \sum_{j=\ell_1}^k \binom{k}{j}\right).$$

Let a be the common threshold state for "0" inputs and let b be the common threshold state for "1" inputs.

(i) If $a=b$ then $\tau(k, i, \ell_0, \ell_1)=0$ if $i < c+d$.

(ii) If $a \neq b$ then $\tau(k, i, \ell_0, \ell_1)=0$ if $i < c$ or $2^k - i < d$.

We are now ready to calculate $\tau(k, i, \ell_0, \ell_1)$. Let

$T(k, i, j, j')$ be the number of mappings on k inputs with $I=i$, $\ell_0 \leq j$ and $\ell_1 \leq j'$. If T is obtained τ may be computed from T via a second order difference, i.e.,

$$\tau = \Delta_{\ell_0, \ell_1}^2(T).$$

More specifically this notation means

$$\begin{aligned} \tau(k, i, \ell_0, \ell_1) &= T(k, i, \ell_0, \ell_1) - T(k, i, \ell_0 - 1, \ell_1) \\ &\quad - T(k, i, \ell_0, \ell_1 - 1) + T(k, i, \ell_0 - 1, \ell_1 - 1). \end{aligned}$$

Theorem 5 enables us to compute $T(k, i, \ell_0, \ell_1)$.

Theorem 5: Let $e_1 = \binom{2^k - (c+d)}{2^k - 1}$, $e_2 = \binom{2^k - (c+d)}{1}$,

$e_3 = \binom{2^k - (c+d)}{i-c}$, and $e_4 = \binom{2^k - (c+d)}{i-d}$ and define

$\binom{b}{a} \equiv 0$ if $a > b$. Then

$$\begin{aligned} T(k, i, \ell_0, \ell_1) &= 2(e_1 + e_2 + e_3 + e_4) \text{ if } 2^{k-1} < i \leq 2^k \\ &= 2(e_1 + e_3) \text{ if } i = 2^{k-1}. \end{aligned}$$

Lastly we may obtain $\tau(k, i, \ell)$.

Theorem 6: $\tau(k, i, \ell) = 2 \sum_{\ell'=\ell+1}^k \tau(k, i, \ell, \ell') + \tau(k, i, \ell, \ell)$.

Finally we turn to the calculation of $\sigma(k, f, \ell)$, the number of mappings on k inputs with exactly f forcing inputs and threshold ℓ . The expression we develop is extremely awkward to calculate. Let $\rho(k, \ell_0, \ell_1, r, s)$, the number of maps on k inputs with thresholds ℓ_0, ℓ_1 respectively and r inputs having forcing state "1", s inputs having forcing state "0". Also let

$R'(k, \ell_0, \ell_1, r, s)$ which is the number of mappings on k inputs having thresholds ℓ_0 and ℓ_1 with at least the first r inputs having forcing state "1" at least the next s inputs having forcing state "0". Theorem 7 calculates R' , Theorem 8 shows how R' may be adjusted to yield p and finally Theorem 9 obtains σ from p .

Theorem 7: Case (i) $r \geq 1, s \geq 1$.

$$\text{If } \ell_0 + \ell_1 = k+2, R'(k, \ell_0, \ell_1, r, s) = 2 \binom{k-(r+s)}{\ell_0-r-1} (2^{-1}).$$

$$\text{If } \ell_0 + \ell_1 > k+2, R'(k, \ell_0, \ell_1, r, s) =$$

$$2 \binom{k-(r+s)}{\ell_0-r-1} (2^{-1}) \binom{k-(r+s)}{\ell_1-s-1} (2^{-1}) 2^{\sum_{i=k-r-\ell_1+2}^{\ell_0-r-2} \binom{k}{i}}.$$

Case (ii) $r \geq 1, s=0$.

$$\text{If } \ell_0 = k, \ell_0 + \ell_1 = k+2, R'(k, k, 2, r, 0) = 2^{k-r+2} - 6.$$

$$\text{If } \ell_0 = k, \ell_0 + \ell_1 > k+2, R'(k, k, \ell_1, r, 0) =$$

$$4 \binom{k-r}{\ell_1-1} (2^{k-r-2} - 1) 2^{\sum_{i=k-r-\ell_1+2}^{k-r-2} \binom{k}{i}}.$$

If $\ell_0 < k$, same expressions as in case (i).

Case (iii) $s \geq 1, r=0$.

$$\text{If } \ell_1 = k, \ell_0 + \ell_1 = k+2, R'(k, 2, k, 0, s) = 2^{k-s+2} - 6. \text{ If } \ell_1 = k,$$

$$\ell_0 + \ell_1 > k+2, R'(k, \ell_0, k, 0, s) =$$

$$\frac{\binom{k-s}{\ell_0-1}}{4(2^{\ell_0-1}-1)(2^{k-s}-1)2} \sum_{i=k-\ell_1+2}^{\ell_0-2} \binom{k}{i}.$$

If $\ell_0 < k$, same expressions as in case (i). Given R' we have

Theorem 8:

$$\rho(k, \ell_0, \ell_1, r, s) = \frac{k!}{r!s!(k-(r+s))!} \sum_{\substack{0 \leq j, j' \\ j+j' \leq k-(r+s)}} (-1)^{j+j'} \\ \cdot \frac{(k-(r+s))!}{j!j'!(k-(r+s)-j-j')!} R'(k, \ell_0, \ell_1, r+j, \\ s+j').$$

Finally using ρ we have

Theorem 9:

$$\sigma(k, f, \ell) = 2 \sum_{\ell'=\ell+1}^k \sum_{r+s=f} \rho(k, \ell, \ell', r, s) + \sum_{r+s=f} \rho(k, \ell, \ell', r, s).$$

4. A REPRESENTATION THEOREM

Representation theorems for mappings have been discussed previously. For example, Rosen discusses the canonical factorization of a noncontractible mapping (absolute threshold $\ell=1$) obtained through subproducts of its domain. Each mapping in a system composed of such mappings is uniquely factored into a composition of

a projection mapping and a nonfactorable mapping. Babcock attempts a representation theorem for forcible mappings. Unfortunately her results are not applicable to the entire class of forcible mappings.

It is our goal then to provide a unique representation of any mapping on k inputs which is forcible on exactly j of them. We shall assume $j \geq 1$ since only a trivial decomposition is possible if $j=0$. By permutation of the input rows, we may consider the first j inputs as the forcing ones. There is no unique permutation which achieves this reordering since we are just effecting a relabeling of the inputs. For convenience we assume the mapping has forcing state "1" and forced value "1". We denote the mapping generically by m and state our factorization theorem.

Theorem 5: If m is forcing on its first j inputs with forcing state "1" and forced value "1" then m may be uniquely factored as $m(x_1, \dots, x_k) = \max(h(x_1, \dots, x_j), g(x_{j+1}, \dots, x_k))$ where

$$h(x_1, \dots, x_j) = \begin{cases} 1 & \sum_{i=1}^j x_i \geq 1 \\ 0 & \sum_{i=1}^j x_i = 0 \end{cases}$$

and $g(x_{j+1}, \dots, x_k) = m(0, 0, \dots, 0, x_{j+1}, \dots, x_k)$.

By this decomposition we have factored the k -dimensional set of inputs into a j -dimensional and a $k-j$ dimensional set. The mapping on the first j coordinates is a projection while the mapping on the remaining $k-j$ coordinates is non-factorable.

The mapping may also be represented as a composition of g and h and the composition is commutative. Let $A = A_1 = \{0, 1\}$ and let $A^k = \prod_{i=1}^k A_i$. Then m is a mapping from $A^k \rightarrow A$. Suppose

$$g_h = \begin{cases} g & \text{if } h = 0 \\ 1 & \text{if } h = 1 \end{cases} \quad \text{and}$$

$$h_g = \begin{cases} h & \text{if } g = 0 \\ 1 & \text{if } g = 1. \end{cases}$$

Then $m = g_h \cdot h = h_g \cdot g$ and Figure 3 illustrates the commutative aspect of this composition.

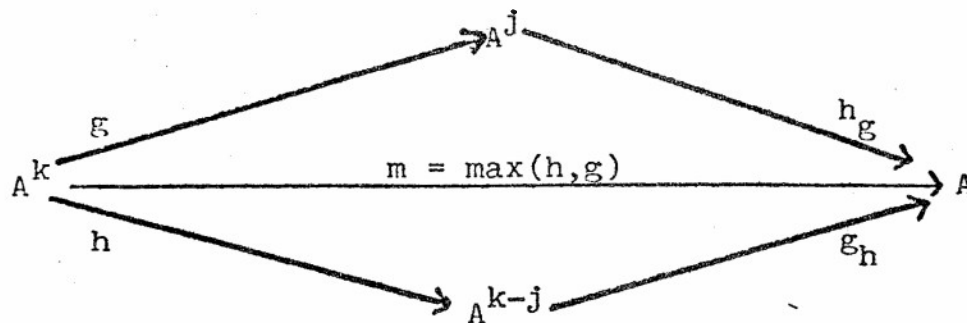


Figure 3: A Graphical Depiction of the Factorization Theorem

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